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A strong convergence theorem for solutions of equilibrium problems and asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense

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Abstract
In this paper, common solutions to an equilibrium problem and a nonlinear operator equation involving a finite family of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense are discussed. Strong convergence theorems of common solutions are established in a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property.
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1 Introduction

Equilibrium problems have been revealed as a very powerful and important tool in the study of nonlinear phenomena. They have turned out to be very useful in studying optimization problems, differential equations, and minimax theorems and in certain applications to mechanics and economic theory; see [1–27] and the references therein. Important practical situations motivate the study of a system of equilibrium problems. For instance, the flow of fluid through a fissured porous medium and certain models of plasticity led to such problems; see, for instance, [28]. Because of their importance, they have been extensively analyzed. The aim of this paper is to present an iterative method for solving solutions of an equilibrium problem, which is known as the Ky Fan inequality, and a nonlinear operator equation involving a finite family of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense.

The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, an iterative algorithm is presented. A strong convergence theorem is established in a reflexive Banach space. Some results in Hilbert spaces are also discussed.

2 Preliminaries

Let E be a real Banach space. Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. Recall that E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|\frac{x_n+y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth if $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U_E$. It is said to be uniformly smooth if the above limit is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that if E is uniformly smooth if and only if E^* is uniformly convex.

Recall that E enjoys the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec-Klee property.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E.$$

Observe that in a Hilbert space H, the equality is reduced to $\phi(x,y) = \|x-y\|^2$, $x, y \in H$. As we all know, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [29] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection P_C in Hilbert spaces. Recall that the generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y\in C}\phi(y, x).$$

Existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$\bigl(\|x\| - \|y\|\bigr)^2 \le \phi(x,y) \le \bigl(\|y\| + \|x\|\bigr)^2, \quad \forall x, y \in E \tag{2.1}$$

and

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E. \tag{2.2}$$

Remark 2.1 If E is a reflexive, strictly convex and smooth Banach space, then $\phi(x,y) = 0$ if and only if $x = y$; for more details, see [29] and the reference therein.

Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers and let $A : C \to E^*$ be a mapping. Consider the following equilibrium problem. Find $p \in C$ such that

$$f(p,q) + \langle Ap, q - p\rangle \ge 0, \quad \forall q \in C. \tag{2.3}$$

We use $EP(f,A)$ to denote the solution set of inequality (2.3). That is,

$$EP(f) = \{p \in C : f(p,q) + \langle Ap, q-p\rangle \geq 0, \forall q \in C\}.$$

If $A = 0$, then problem (2.3) is reduced to the following Ky Fan inequality. Find $p \in C$ such that

$$f(p,q) \geq 0, \quad \forall q \in C. \tag{2.4}$$

We use $EP(f)$ to denote the solution set of inequality (2.4). That is,

$$EP(f) = \{p \in C : f(p,q) \geq 0, \forall q \in C\}.$$

If $f = 0$, then problem (2.3) is reduced to the classical variational inequality. Find $p \in C$ such that

$$\langle Ap, q-p\rangle \geq 0, \quad \forall q \in C. \tag{2.5}$$

We use $VI(C,A)$ to denote the solution set of inequality (2.5). That is,

$$VI(C,A) = \{p \in C : \langle Ap, q-p\rangle \geq 0, \forall q \in C\}.$$

Recall that a mapping $A : C \to E^*$ is said to be α-inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2.$$

For solving problem (2.3), let us assume that the nonlinear mapping $A : C \to E^*$ is α-inverse-strongly monotone and the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$, $\forall x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$, $\forall x,y \in C$;

(A3)

$$\limsup_{t \downarrow 0} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y), \quad \forall x,y,z \in C;$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.

Let C be a nonempty subset of E and $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T. T is said to be asymptotically regular on C if for any bounded subset K of C,

$$\limsup_{n \to \infty} \{\|T^{n+1}x - T^n x\| : x \in K\} = 0.$$

T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Recall that a point p in C is said to be an asymptotic fixed point of T iff C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

A mapping T is said to be relatively nonexpansive iff

$$\widetilde{F}(T) = F(T) \neq \emptyset, \qquad \phi(p, Tx) \leq \phi(p,x), \quad \forall x \in C, \forall p \in F(T).$$

A mapping T is said to be relatively asymptotically nonexpansive iff

$$\widetilde{F}(T) = F(T) \neq \emptyset, \qquad \phi\bigl(p, T^n x\bigr) \leq (1 + \mu_n)\phi(p,x), \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1,$$

where $\{\mu_n\} \subset [0,\infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

Remark 2.2 The class of relatively asymptotically nonexpansive mappings was first considered in Agarwal *et al.* [30].

Recall that a mapping T is said to be quasi-ϕ-nonexpansive iff

$$F(T) \neq \emptyset, \qquad \phi(p, Tx) \leq \phi(p,x), \quad \forall x \in C, \forall p \in F(T).$$

Recall that a mapping T is said to be asymptotically quasi-ϕ-nonexpansive iff there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$F(T) \neq \emptyset, \qquad \phi\bigl(p, T^n x\bigr) \leq (1 + \mu_n)\phi(p,x), \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1.$$

Remark 2.3 The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive mappings do not require the restriction $F(T) = \widetilde{F}(T)$; for more details, see [31–33] the reference therein.

Remark 2.4 The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are generalizations of the class of quasi-nonexpansive mappings and the class of asymptotically quasi-nonexpansive mappings in Banach spaces.

Recall that T is said to be asymptotically quasi-ϕ-nonexpansive in the intermediate sense iff $F(T) \neq \emptyset$ and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{p\in F(T), x\in C} \bigl(\phi\bigl(p, T^n x\bigr) - \phi(p,x)\bigr) \leq 0. \tag{2.6}$$

Put

$$\xi_n = \max\Bigl\{0, \sup_{p\in F(T), x\in C} \bigl(\phi\bigl(p, T^n x\bigr) - \phi(p,x)\bigr)\Bigr\}.$$

It follows that $\xi_n \to 0$ as $n \to \infty$. Then (2.6) is reduced to the following:

$$\phi\bigl(p, T^n x\bigr) \leq \phi(p,x) + \xi_n, \quad \forall p \in F(T), \forall x \in C. \tag{2.7}$$

Remark 2.5 The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense was first considered by Qin and Wang [34]; see also [35].

Remark 2.6 The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense is a generalization of the class of asymptotically quasi-nonexpansive mappings in the intermediate sense, which was considered by Kirk [36] in the framework of Banach spaces.

In order to state our main results, we also need the following lemmas.

Lemma 2.7 [29] *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.8 [29] *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.9 [37] *Let E be a smooth, strictly convex, and reflexive Banach space, and let C be a nonempty closed convex subset of E. Let $A : C \to E^*$ be an α-inverse-strongly monotone mapping and f be a bifunction satisfying conditions* (A1)-(A4). *Let $r > 0$ be any given number, and let $x \in E$ define a mapping $K_r : C \to C$ as follows: for any $x \in C$,*

$$K_r x = \left\{ p \in C : f(p, q) + \langle Ap, q - p \rangle + \frac{1}{r} \langle q - p, Jp - Jx \rangle \geq 0 \right\}, \quad \forall q \in C.$$

Then the following conclusions hold:

(1) *K_r is single-valued;*

(2) *K_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle K_r x - K_r y, JK_r x - JK_r y \rangle \leq \langle S_r x - S_r y, Jx - Jy \rangle;$$

(3) $F(K_r) = EP(f, A)$;

(4) *K_r is quasi-ϕ-nonexpansive;*

(5)

$$\phi(q, K_r x) + \phi(K_r x, x) \leq \phi(q, x), \quad \forall q \in F(K_r);$$

(6) *$EP(f, A)$ is closed and convex.*

Lemma 2.10 [38] *Let E be a smooth and uniformly convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\left\| tx + (1 - t)y \right\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g\left(\|x - y\|\right)$$

for all $x, y \in B_r = \{x \in E : \|x\| \leq r\}$ and $t \in [0, 1]$.

3 Main results

Theorem 3.1 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, and let C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let N be some positive integer. Let $A : C \to E^*$ be a κ_i-inverse-strongly monotone mapping. Let $T_i : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense for every $1 \le i \le N$. Assume that T_i is closed asymptotically regular on C and $\bigcap_{i=1}^N F(T_i) \cap EP(f,A)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_{n,0}Jx_n + \sum_{i=1}^N \alpha_{n,i}JT_i^n x_n), \\ u_n \in C \text{ such that } f(u_n,y) + \langle Au_n + y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\ H_n = \{z \in C : \phi(z,u_n) \le \phi(z,x_n) + N\xi_n\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \end{cases}$$

where $\xi_n = \max\{0, \sup_{p \in \bigcap_{i=1}^N F(T_i), x \in C}(\phi(p, T_i^n x) - \phi(p,x))\}$, $\{\alpha_{n,i}\}$ *is a real number sequence in* $(0,1)$ *for every* $1 \le i \le N$, $\{r_n\}$ *is a real number sequence in* $[k,\infty)$, *where k is some positive real number. Assume that* $\sum_{i=0}^N \alpha_{n,i} = 1$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for every* $1 \le i \le N$. *Then the sequence $\{x_n\}$ converges strongly to* $\Pi_{\bigcap_{i=1}^N F(T_i) \cap EP(f,A)} x_0$, *where* $\Pi_{\bigcap_{i=1}^N F(T_i) \cap EP(f,A)}$ *is the generalized projection from E onto* $\bigcap_{i=1}^N F(T_i) \cap EP(f,A)$.

Proof Since F_i is closed and convex for every $1 \le i \le N$, we obtain from Lemma 2.9 that the common element set $\bigcap_{i=1}^N F(T_i) \cap EP(f,A)$ is closed and convex. Next, we show that both H_n and W_n are closed and convex. From the definition of H_n and W_n, it is obvious that H_n is closed and W_n is closed and convex. We show that H_n is convex since $\phi(z,u_n) \le \phi(z,x_n) + N\xi_n$ is equivalent to

$$2\langle z, Jx_n - Ju_n\rangle \le \|x_n\|^2 - \|u_n\|^2 + N\xi_n.$$

It follows that H_n is convex. This in turn shows that $\Pi_{H_n \cap W_n} x_0$ is well defined. Putting $u_n = k_{r_n} y_n$, from Lemma 2.9 we see that K_{r_n} is quasi-ϕ-nonexpansive. Now, we are in a position to prove that $\bigcap_{i=1}^N F(T_i) \cap EP(f,A) \subset H_n \cap W_n$. Let $w \in \bigcap_{i=1}^N F(T_i) \cap EP(f,A)$,

$$\begin{aligned} \phi(w,u_n) &= \phi(w, K_{r_n} y_n) \\ &\le \phi(w,y_n) \\ &= \phi\left(w, J^{-1}\left(\alpha_{n,0}Jx_n + \sum_{i=1}^N \alpha_{n,i}JT_i^n x_n\right)\right) \\ &= \|w\|^2 - 2\left\langle w, \alpha_{h,0}Jx_n + \sum_{i=1}^N \alpha_{n,i}JT_i^n x_n\right\rangle + \left\|\alpha_{n,0}Jx_n + \sum_{i=1}^N \alpha_{n,i}JT_i^n x_n\right\|^2 \\ &\le \|w\|^2 - 2\alpha_{n,0}\langle w, Jx_n\rangle - 2\sum_{i=1}^N \alpha_{n,i}\langle w, JT_i^n x_n\rangle + \alpha_{n,0}\|x_n\|^2 + \sum_{i=1}^N \alpha_{n,i}\|T_i^n x_n\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \alpha_{n,0}\phi(w,x_n) + \sum_{i=1}^{N} \alpha_{n,i}\phi\bigl(w, T_i^n x_n\bigr) \\
&\leq \alpha_{n,0}\phi(w,x_n) + \sum_{i=1}^{N} \alpha_{n,i}\phi(w,x_n) + \sum_{i=1}^{N} \alpha_{n,i}\xi_n \\
&= \phi(w,x_n) + \sum_{i=1}^{N} \alpha_{n,i}\xi_n \\
&\leq \phi(w,x_n) + N\xi_n.
\end{aligned}
\tag{3.1}
$$

We have $w \in H_n$. This implies that $\bigcap_{i=1}^N F(T_i) \cap EP(f,A) \subset H_n$. On the other hand, we see that $\bigcap_{i=1}^N F(T_i) \cap EP(f,A) \subset H_0 \cap W_0$. Suppose that $\bigcap_{i=1}^N F(T_i) \cap EP(f,A) \subset H_m \cap W_m$ for some m. There exists an element $x_{m+1} \in H_m \cap W_m$ such that $x_{m+1} = \Pi_{H_m \cap W_m} x_0$. In view of Lemma 2.7, we find that

$$\langle x_{m+1} - w, Jx_0 - Jx_{m+1}\rangle \geq 0, \quad w \in H_m \cap W_m.$$

Since $\bigcap_{i=1}^N F(T_i) \cap EP(f,A) \subset H_m \cap W_m$, we arrive at

$$\langle x_{m+1} - w, Jx_0 - Jx_{m+1}\rangle \geq 0 \tag{3.2}$$

for every $w \in \bigcap_{i=1}^N F(T_i) \cap EP(f,A)$. We therefore find that $\bigcap_{i=1}^N F(T_i) \cap EP(f,A) \subset W_{m+1}$. It follows that $\bigcap_{i=1}^N F(T_i) \cap EP(f,A) \subset H_{m+1} \cap W_{m+1}$. This shows that the sequence $\{x_n\}$ is well defined.

Next, we prove that the sequence $\{x_n\}$ is bounded. It follows from the definition of W_n and Lemma 2.7 that $x_n = \Pi_{W_n} x_0$. In view of Lemma 2.8, we find that

$$\phi(x_n,x_0) = \phi(\Pi_{W_n} x_0, x_0) \leq \phi(w,x_0) - \phi(w,x_n) \leq \phi(w,x_0)$$

for each $w \in \bigcap_{i=1}^N F(T_i) \cap EP(f,A) \subset W_n$. This implies that $\{\phi(x_n,x_0)\}$ is bounded. It follows from (2.1) that $\{x_n\}$ is also bounded. Since $x_{n+1} = \Pi_{H_n \cap W_n} x_0 \in W_n$, we find from Lemma 2.7 that $\phi(x_n,x_0) \leq \phi(x_{n+1},x_0)$. Therefore, we obtain that $\{\phi(x_n,x_0)\}$ is nondecreasing. So there exists the limit of $\phi(x_n,x_0)$. It follows from Lemma 2.8 that

$$
\begin{aligned}
\phi(x_{n+1},x_n) &= \phi(x_{n+1}, \Pi_{W_n} x_0) \\
&\leq \phi(x_{n+1},x_0) - \phi(\Pi_{W_n} x_0, x_0) \\
&= \phi(x_{n+1},x_0) - \phi(x_n,x_0).
\end{aligned}
$$

This shows that $\lim_{n\to\infty} \phi(x_{n+1},x_n) = 0$. Since $x_{n+1} = \Pi_{H_n \cap W_n} x_0 \in H_n$, we find that $\lim_{n\to\infty} \phi(x_{n+1},u_n) = 0$. Since the space is reflexive, we may assume, without loss of generality, that $x_n \rightharpoonup \bar{x}$. Since W_n is closed and convex, we find that $\bar{x} \in W_n$. This implies from $x_n = \Pi_{W_n} x_0$ that $\phi(x_n,x_0) \leq \phi(\bar{x},x_0)$. On the other hand, we see from the weakly lower semicontinuity of $\|\cdot\|$ that

$$
\begin{aligned}
\phi(\bar{x},x_0) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx_0\rangle + \|x_0\|^2 \\
&\leq \liminf_{n\to\infty} \bigl(\|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2\bigr)
\end{aligned}
$$

$$
\begin{aligned}
&= \liminf_{n\to\infty} \phi(x_n, x_0) \\
&\le \limsup_{n\to\infty} \phi(x_n, x_0) \\
&\le \phi(\bar{x}, x_0),
\end{aligned}
$$

which implies that $\lim_{n\to\infty} \phi(x_n, x_0) = \phi(\bar{x}, x_0)$. Hence, we have $\lim_{n\to\infty} \|x_n\| = \|\bar{x}\|$. In view of the Kadec-Klee property of E, we find that $x_n \to \bar{x}$ as $n \to \infty$. In view of (2.1), we see that $\lim_{n\to\infty}(\|x_{n+1}\| - \|u_n\|) = 0$. This implies that $\lim_{n\to\infty} \|u_n\| = \|\bar{x}\|$. That is,

$$
\lim_{n\to\infty} \|Ju_n\| = \lim_{n\to\infty} \|u_n\| = \|J\bar{x}\|. \tag{3.3}
$$

This implies that $\{Ju_n\}$ is bounded. Note that both E and E^* are reflexive. We may assume, without loss of generality, that $Ju_n \rightharpoonup u^* \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $u \in E$ such that $Ju = u^*$. It follows that

$$
\begin{aligned}
\phi(x_{n+1}, u_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|u_n\|^2 \\
&= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|Ju_n\|^2.
\end{aligned}
$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above yields that

$$
\begin{aligned}
0 &\ge \|\bar{x}\|^2 - 2\langle \bar{x}, u^*\rangle + \|u^*\|^2 \\
&= \|\bar{x}\|^2 - 2\langle \bar{x}, Ju\rangle + \|Ju\|^2 \\
&= \|\bar{x}\|^2 - 2\langle \bar{x}, Ju\rangle + \|u\|^2 \\
&= \phi(\bar{x}, u).
\end{aligned}
$$

That is, $\bar{x} = u$, which in turn implies that $u^* = J\bar{x}$. It follows that $Ju_n \rightharpoonup J\bar{x} \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain from (3.3) that $\lim_{n\to\infty} Ju_n = J\bar{x}$. Since $J^{-1}: E^* \to E$ is demi-continuous and E enjoys the Kadec-Klee property, we obtain that $u_n \to \bar{x}$, as $n \to \infty$. Note that $\|x_n - u_n\| \le \|x_n - \bar{x}\| + \|\bar{x} - u_n\|$. It follows that

$$
\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.4}
$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$
\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.5}
$$

On the other hand, we have

$$
\begin{aligned}
\phi(w, x_n) - \phi(w, u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle w, Jx_n - Ju_n\rangle \\
&\le \|x_n - u_n\|\bigl(\|x_n\| + \|u_n\|\bigr) + 2\|w\|\|Jx_n - Ju_n\|.
\end{aligned}
$$

We, therefore, find that

$$
\lim_{n\to\infty} \bigl(\phi(w, x_n) - \phi(w, u_n)\bigr) = 0. \tag{3.6}
$$

Since E is uniformly smooth, we know that E^* is uniformly convex. In view of Lemma 2.10, we find that

$$\begin{aligned}
\phi(w,u_n) &= \phi(w,K_{r_n}y_n)\\
&\le \phi(w,y_n)\\
&= \phi\left(w,J^{-1}\left(\alpha_{n,0}Jx_n+\sum_{i=1}^N\alpha_{n,i}JT_i^nx_n\right)\right)\\
&= \|w\|^2-2\left\langle w,\alpha_{n,0}Jx_n+\sum_{i=1}^N\alpha_{n,i}JT_i^nx_n\right\rangle+\left\|\alpha_{n,0}Jx_n+\sum_{i=1}^N\alpha_{n,i}JT_i^nx_n\right\|^2\\
&\le \|w\|^2-2\alpha_{n,0}\langle w,Jx_n\rangle-2\sum_{i=1}^N\alpha_{n,i}\langle w,JT_i^nx_n\rangle+\alpha_{n,0}\|x_n\|^2\\
&\quad+\sum_{i=1}^N\alpha_{n,i}\|T_i^nx_n\|^2-\alpha_{n,0}\alpha_{n,1}g(\|Jx_n-JT_1^nx_n\|)\\
&= \alpha_{n,0}\phi(w,x_n)+\sum_{i=1}^N\alpha_{n,i}\phi(w,T_i^nx_n)-\alpha_{n,0}\alpha_{n,1}g(\|Jx_n-JT_1^nx_n\|)\\
&\le \alpha_{n,0}\phi(w,x_n)+\sum_{i=1}^N\alpha_{n,i}\phi(w,x_n)+\sum_{i=1}^N\alpha_{n,i}\xi_h\\
&\quad-\alpha_{n,0}\alpha_{n,1}g(\|Jx_n-JT_1^nx_n\|)\\
&= \phi(w,x_n)+\sum_{i=1}^N\alpha_{n,i}\xi_n-\alpha_{n,0}\alpha_{n,1}g(\|Jx_n-JT_1^nx_n\|)\\
&\le \phi(w,x_n)+N\xi_n-\alpha_{n,0}\alpha_{n,1}g(\|Jx_n-JT_1^nx_n\|).
\end{aligned}$$

It follows that $\alpha_{n,0}\alpha_{n,1}g(\|Jx_n-JT_1^nx_n\|)\le\phi(w,x_n)-\phi(w,u_n)+\xi_n$. In view of the restriction on the sequences, we find from (3.6) that $\lim_{n\to\infty}g(\|Jx_n-JT_1^nx_n\|)=0$. It follows that $\lim_{n\to\infty}\|Jx_n-JT_1^nx_n\|=0$. In the same way, we obtain that $\lim_{n\to\infty}\|Jx_n-JT_i^nx_n\|=0$, $\forall 2\le i\le N$. Notice that $\|JT_i^nx_n-J\bar{x}\|\le\|JT_i^nx_n-Jx_n\|+\|Jx_n-J\bar{x}\|$. It follows that $\lim_{n\to\infty}\|JT_i^nx_n-J\bar{x}\|=0$. The demicontinuity of $J^{-1}:E^*\to E$ implies that $T_i^nx_n\rightharpoonup\bar{x}$. Note that

$$\big|\|T_i^nx_n\|-\|\bar{x}\|\big|=\big|\|JT_i^nx_n\|-\|J\bar{x}\|\big|\le\|JT_i^nx_n-J\bar{x}\|.$$

This implies that $\lim_{n\to\infty}\|T_i^nx_n\|=\|\bar{x}\|$. Since E has the Kadec-Klee property, we obtain that $\lim_{n\to\infty}\|T_i^nx_n-\bar{x}\|=0$. On the other hand, we have

$$\|T_i^{n+1}x_n-\bar{x}\|\le\|T_i^{n+1}x_n-T_i^nx_n\|+\|T_i^nx_n-\bar{x}\|.$$

It follows from the asymptotic regularity of T_i that $\lim_{n\to\infty}\|T_i^{n+1}x_n-\bar{x}\|=0$. That is, $T_iT_i^nx_n\to\bar{x}$. From the closedness of T_i, we find $\bar{x}=T_i\bar{x}$ for every $1\le i\le N$. This proves $\bar{x}\in\bigcap_{i=1}^N F(T_i)$. Now, we state that $\bar{x}\in EP(f,A)$. In view of Lemma 2.9, we find that

$$\phi(u_n,y_n)\le\phi(w,y_n)-\phi(w,u_n)\le\phi(w,x_n)+N\xi_n-\phi(w,u_n).$$

It follows from (3.6) that $\lim_{n\to\infty}\phi(u_n,y_n)=0$. This implies that $\lim_{n\to\infty}(\|u_n\|-\|y_n\|)=0$. It follows from (3.4) that $\lim_{n\to\infty}\|y_n\|=\|\bar{x}\|$. It follows that

$$\lim_{n\to\infty}\|Jy_n\|=\lim_{n\to\infty}\|y_n\|=\|\bar{x}\|=\|J\bar{x}\|.$$

This shows that $\{Jy_n\}$ is bounded. Since E^* is reflexive, we may assume that $Jy_n \rightharpoonup y^* \in E^*$. In view of $J(E)=E^*$, we see that there exists $y\in E$ such that $Jy=y^*$. It follows that $\phi(u_n,y_n)=\|u_n\|^2-2\langle u_n,Jy_n\rangle+\|Jy_n\|^2$. Taking $\liminf_{n\to\infty}$ on the both sides of the equality above yields that

$$\begin{aligned}0&\geq\|\bar{x}\|^2-2\langle\bar{x},y^*\rangle+\|y^*\|^2\\&=\|\bar{x}\|^2-2\langle\bar{x},Jy\rangle+\|Jy\|^2\\&=\|\bar{x}\|^2-2\langle\bar{x},Jy\rangle+\|y\|^2\\&=\phi(\bar{x},y).\end{aligned}$$

That is, $\bar{x}=y$, which in turn implies that $y^*=J\bar{x}$. It follows that $Jy_n\rightharpoonup J\bar{x}\in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain that $Jy_n-J\bar{x}\to 0$ as $n\to\infty$. Note that $J^{-1}:E^*\to E$ is demi-continuous. It follows that $y_n\rightharpoonup\bar{x}$. Since E enjoys the Kadec-Klee property, we obtain that $y_n\to\bar{x}$ as $n\to\infty$. Note that $\|u_n-y_n\|\leq\|u_n-\bar{x}\|+\|\bar{x}-y_n\|$. This implies that $\lim_{n\to\infty}\|u_n-y_n\|=0$. Since J is uniformly norm-to-norm continuous on any bounded sets, we have $\lim_{n\to\infty}\|Ju_n-Jy_n\|=0$. In view of the assumption $r_n\geq k$, we see that

$$\lim_{n\to\infty}\frac{\|Ju_n-Jy_n\|}{r_n}=0. \tag{3.7}$$

Since $u_n=K_{r_n}y_n$, we find that

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,Ju_n-Jy_n\rangle\geq 0,\quad\forall y\in C,$$

where $F(u_n,y)=f(u_n,y)+\langle Au_n+y-u_n\rangle$ for every $y\in C$. It follows from (A2) that

$$\|y-u_n\|\frac{\|Ju_n-Jy_n\|}{r_n}\geq\frac{1}{r_n}\langle y-u_n,Ju_n-Jy_n\rangle\geq F(y,u_n),\quad\forall y\in C.$$

It follows from (3.7) that

$$F(y,\bar{x})\leq 0,\quad\forall y\in C.$$

For $0<t<1$ and $y\in C$, define $y_t=ty+(1-t)\bar{x}$. It follows that $y_t\in C$, which yields that $F(y_t,\bar{x})\leq 0$. It follows from (A1) and (A4) that

$$0=F(y_t,y_t)\leq tF(y_t,y)+(1-t)F(y_t,\bar{x})\leq tF(y_t,y).$$

That is,

$$F(y_t,y)\geq 0.$$

Letting $t \downarrow 0$, we obtain from (A3) that $F(\bar{x}, y) \geq 0$, $\forall y \in C$. That is, $f(u_n, y) + \langle Au_n + y - u_n \rangle \geq 0$ for every $y \in C$. This implies that $\bar{x} \in EP(f, A)$. This completes the proof that $\bar{x} \in \bigcap_{i=1}^N F(T_i) \cap EP(f, A)$. Since $x_{n+1} = \Pi_{H_n \cap W_n} x_0$. In view of Lemma 2.7, we find that

$$\langle x_{n+1} - w, Jx_0 - Jx_{n+1} \rangle \geq 0, \quad w \in H_n \cap W_n.$$

Since $\bigcap_{i=1}^N F(T_i) \cap EP(f, A) \subset H_n \cap W_n$, we arrive at

$$\langle x_{n+1} - w, Jx_0 - Jx_{n+1} \rangle \geq 0.$$

Letting $n \to \infty$ in the above inequality, we obtain that

$$\langle \bar{x} - w, Jx_0 - J\bar{x} \rangle \geq 0, \quad \forall w \in \bigcap_{i=1}^N F(T_i) \cap EP(f, A).$$

It follows from Lemma 2.7 that $\bar{x} = \Pi_{\bigcap_{i=1}^N F(T_i) \cap EP(f,A)} x_0$. This completes the proof. □

Remark 3.2 Theorem 3.1 includes the corresponding results in the literature as special cases. Since every uniformly convex Banach space enjoys the Kadec-Klee property, the framework of the space can be applicable to L^p, $p \geq 1$.

Next, we state a result on Ky Fan inequality (2.4) and a single asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense.

Corollary 3.3 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, and let C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $T : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense. Assume that T is closed asymptotically regular on C and $F(T) \cap EP(f)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT^n x_n), \\ u_n \in C \textit{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C, \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n) + N\xi_n\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \end{cases}$$

where $\xi_n = \max\{0, \sup_{p \in F(T), x \in C}(\phi(p, T^n x) - \phi(p, x))\}$, $\{\alpha_n\}$ is a real number sequence in $(0, 1)$, $\{r_n\}$ is a real number sequence in $[k, \infty)$, where k is some positive real number. Assume that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T) \cap EP(f)} x_0$, where $\Pi_{F(T) \cap EP(f)}$ is the generalized projection from E onto $F(T) \cap EP(f)$.

If the mapping T is quasi-ϕ-nonexpansive, we find from Corollary 3.3 the following.

Corollary 3.4 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, and let C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $T : C \to C$ be a quasi-ϕ-nonexpansive mapping. Assume that $F(T) \cap EP(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \end{cases}$$

where $\{\alpha_n\}$ is a real number sequence in $(0,1)$, $\{r_n\}$ is a real number sequence in $[k,\infty)$, where k is some positive real number. Assume that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap EP(f)} x_0$, where $\Pi_{F(T)\cap EP(f)}$ is the generalized projection from E onto $F(T) \cap EP(f)$.

Finally, we give a result in the framework of Hilbert spaces based on Theorem 3.1.

Corollary 3.5 *Let E be a Hilbert space, and let C be a nonempty closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let N be some positive integer. Let $A : C \to E$ be a κ_i-inverse-strongly monotone mapping. Let $T_i : C \to C$ be an asymptotically quasi-nonexpansive mapping in the intermediate sense for every $1 \leq i \leq N$. Assume that T_i is closed asymptotically regular on C and $\bigcap_{i=1}^N F(T_i) \cap EP(f,A)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_{n,0} x_n + \sum_{i=1}^N \alpha_{n,i} T_i^n x_n, \\ u_n \in C \text{ such that } f(u_n, y) + \langle Au_n + y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \quad \forall y \in C, \\ H_n = \{z \in C : \|z - u_n\|^2 \leq \|z - x_n\|^2 + N\xi_n\}, \\ W_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = \operatorname{Proj}_{H_n \cap W_n} x_0, \end{cases}$$

where $\xi_n = \max\{0, \sup_{p \in \bigcap_{i=1}^N F(T_i), x \in C}(\|p - T_i^n x\|^2 - \|p - x\|^2)\}$, $\{\alpha_{n,i}\}$ is a real number sequence in $(0,1)$ for every $1 \leq i \leq N$, $\{r_n\}$ is a real number sequence in $[k,\infty)$, where k is some positive real number. Assume that $\sum_{i=0}^N \alpha_{n,i} = 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for every $1 \leq i \leq N$. Then the sequence $\{x_n\}$ converges strongly to $\operatorname{Proj}_{\bigcap_{i=1}^N F(T_i)\cap EP(f,A)} x_0$, where $\operatorname{Proj}_{\bigcap_{i=1}^N F(T_i)\cap EP(f,A)}$ is the metric projection from E onto $\bigcap_{i=1}^N F(T_i) \cap EP(f,A)$.

Proof Since $\phi(x,y) = \|x - y\|^2$ and $J = I$ in the framework of Hilbert spaces, we draw the desired conclusion immediately. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to this manuscript. Both authors read and approved the final manuscript.

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